ISRAEL JOURNAL OF MATHEMATICS, Vol 27, No 2, 1977

RATIONAL ERGODICITY AND A METRIC INVARIANT FOR MARKOV SHIFTS

BY
JON AARONSON

ABSTRACT

The concept of rational ergodicity is introduced and used to construct a metric invariant for the class of rationally ergodic transformations (which includes all ergodic Markov shifts).

§0. Introduction

We study invertible ergodic measure preserving transformations (i.e.m.p.t.s) of σ-finite (usually infinite) measure spaces; (the assumption of invertibility, made for conciseness, is not essential, except in §6).

Rational ergodicity is a ratio limit property. We discuss various ratio limit properties of i.e.m.p.t.s in §1, before defining: "weak rational ergodicity" and "rational ergodicity", and the "return sequence" and "asymptotic type" associated with a rationally ergodic transformation.

In §2, we define some metric relationships between m.p.t.s. Asymptotic type is a metric invariant for rationally ergodic transformations, which, when restricted to ergodic Markov shifts (shown to be rationally ergodic in §3), refines the metric invariants of Rudolfer ([23]).

We combine the concepts of return sequence and entropy (introduced by Krengel in [18]) to construct, in §4, a still finer metric invariant (normalised asymptotic type) for rationally ergodic transformations. We construct an uncountable collection of ergodic Markov shifts (preserving infinite measure), with the same asymptotic type, but different normalised asymptotic types. When restricted to e.m.p.t.s of finite measure spaces: normalised asymptotic type boils down to Kolmogorov–Sinai entropy (cf. [3]).

In §5, we study the metric theory of random walks on the integers, in terms of their jump distributions. We construct an uncountable collection of dissimilar, ergodic random walks. This collection could be separated by the invariants of

Received June 28, 1976

Rudolfer. We also prove that the variances of the jump distributions of two similar (see §2) ergodic random walks are simultaneously finite or infinite, for this we need the concept of asymptotic type.

We examine in §6 other ratio limit properties of i.e.m.p.t.s, which, although not relevant to the development of the results of earlier sections, may be of interest in themselves.

This work was done at the Hebrew University of Jerusalem under the supervision of Prof. Benjamin Weiss, to whom the author is most grateful for helpful conversations, excellent supervision and much encouragement.

§1. Rational ergodicity and other ratio limit properties

Let $(X, \mathcal{B}, \mu, T)$ be an i.e.m.p.t. with $\mu(X) \leqq \infty$. Let $\mathcal{F} = \{A \in \mathcal{B} : 0 < \mu(A) < \infty\}$ and, for $A \in \mathcal{F}$, let: $\mathcal{B} \cap A = \{B \in \mathcal{B} : B \subseteq A\}$, $a_n(A) = a_n(A, T) = \sum_{k=0}^{n} \mu(A \cap T^{-k}A)$.

If $\mu(X) < \infty$ then the ergodic theorem implies that

$$(1.1) \qquad \sum_{k=0}^{n} \mu(B \cap T^{-k}C) \sim n \frac{\mu(B)\mu(C)}{\mu(X)} \qquad \text{as } n \to \infty \qquad \forall B, C \in \mathcal{F}.$$

This, in turn, implies that

$$(1.2) \qquad \frac{\sum_{k=0}^{n} \mu(A \cap T^{-k}B)}{\sum_{k=0}^{n} \mu(C \cap T^{-k}D)} \xrightarrow[n\to\infty]{} \frac{\mu(A)\mu(B)}{\mu(C)\mu(D)} \qquad \forall A, B, C, D$$

— a condition that can at least be stated when $\mu(X) = \infty$, even though in this case (by the ergodic theorem)

$$(1.3) \qquad \sum_{k=0}^{n} \mu(B \cap T^{-k}C) = o(n) \qquad \text{as } n \to \infty.$$

In fact, as will be proven in §6, a necessary condition for (1.2) is that $\mu(X) < \infty$. Nevertheless, we look for properties with the flavour of (1.2), but which are satisfied by some i.e.m.p.t. of an infinite measure space.

One such property is

$$(1.4) \qquad \frac{\sum_{k=0}^{n} \mu(A \cap T^{-k}B)}{\sum_{k=0}^{n} \mu(C \cap T^{-k}D)} \xrightarrow[n\to\infty]{} \frac{\mu(A)\mu(B)}{\mu(C)\mu(D)} \qquad \forall A, B, C, D \in \mathcal{C}$$

where $\mathcal{C}$ is μ-dense in $\mathcal{F}$.

We will see in §6 that every i.e.m.p.t. of a separable space satisfies (1.4) for many different μ-dense subcollections $\mathscr{C}$ of $\mathscr{F}$.

In this section, we will examine the property

$$(1.5)\qquad \frac{1}{a_n(A)}\sum_{k=0}^{n}\mu(B\cap T^{-k}C)\xrightarrow[n\to\infty]{}\frac{\mu(B)\mu(C)}{\mu(A)^2}\qquad \forall B,C\in\mathscr{B}\cap A$$

which will be seen in Proposition 1.1 to imply a stronger version of (1.4).

It is evident that (1.5) is a property of the set $A\in\mathscr{F}$ and, accordingly, we let $R(T)$ denote the collection of sets in $\mathscr{F}$ satisfying (1.5).

PROPOSITION 1.1. *Let T be an i.e.m.p.t. and let $A\in\mathscr{F}$, then the following are equivalent*:

(i) $A\in R(T)$,

(ii) $$\frac{1}{a_n(A)}\sum_{k=0}^{n}\mu(B\cap T^{-k}C)\xrightarrow[n\to\infty]{}\frac{\mu(B)\mu(C)}{\mu(A)^2}$$

$$(1.6)\qquad \forall B,C\in\mathscr{F}_A=\bigcup_{n=0}^{\infty}\mathscr{B}\cap\bigcup_{k=0}^{n}T^{-k}A,$$

(iii) $$\underline{\lim}_{n\to\infty}\frac{1}{a_n(A)}\sum_{k=0}^{n}\mu(B\cap T^{-k}C)\geqq\frac{\mu(B)\mu(C)}{\mu(A)^2}$$

(1.7) $\forall B,C\in\mathscr{F}$.

PROOF. (i) ⇒ (ii). Let $B,C\in\mathscr{F}_A$ and write

$$B=\bigcup_{k=0}^{M}B_k\ \text{(disj)},\qquad C=\bigcup_{l=0}^{N}C_l\ \text{(disj)}\qquad \text{where } T^kB_k,\ T^lC_l\subseteq A.$$

Now, $\forall k,l$

$$\frac{1}{a_n(A)}\sum_{j=0}^{n}\mu(B_k\cap T^{-j}C_l)=\frac{1}{a_n(A)}\sum_{j=0}^{n}\mu(T^kB_k\cap T^{-j-l+k}(T^lC_l))$$

$$=\frac{1}{a_n(A)}\sum_{j=l-k}^{n+l-k}\mu(T^kB_k\cap T^{-j}(T^lC_l))\xrightarrow[n\to\infty]{}\frac{\mu(B_k)\mu(C_l)}{\mu(A)^2}$$

$$\because\ T^kB_k,\ T^lC_l\subseteq A.$$

Hence

$$\frac{1}{a_n(A)}\sum_{j=0}^{n}\mu(B\cap T^{-j}C)=\sum_{k=0}^{M}\sum_{l=0}^{N}\frac{1}{a_n(A)}\sum_{j=0}^{n}\mu(B_k\cap T^{-j}C_l)$$

$$\xrightarrow[n\to\infty]{}\sum_{k=0}^{M}\sum_{l=0}^{N}\frac{\mu(B_k)\mu(C_l)}{\mu(A)^2}=\frac{\mu(B)\mu(C)}{\mu(A)^2}.$$

(ii) $\Rightarrow$ (iii). Let $B, C \in \mathscr{F}$. Since T is ergodic, and $\mu(A)>0$, $\bigcup_{n=0}^{\infty} T^{-n}A = X$ (mod μ) and so: $\forall \varepsilon>0$ $\exists B'_\varepsilon, C'_\varepsilon \in \mathscr{F}_A$ s.t. $B'_\varepsilon \subseteq B$, $C'_\varepsilon \subseteq C$ and $\mu(B'_\varepsilon) > \mu(B)-\varepsilon$, $\mu(C'_\varepsilon) > \mu(C) - \varepsilon$. Hence

$$\lim_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}C) \geqq \lim_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B'_\varepsilon \cap T^{-k}C'_\varepsilon)$$

$$= \frac{\mu(B'_\varepsilon)\mu(C'_\varepsilon)}{\mu(A)^2} \geqq \frac{(\mu(B)-\varepsilon)(\mu(C)-\varepsilon)}{\mu(A)^2} \qquad \forall \varepsilon > 0.$$

(iii) $\Rightarrow$ (i). Let $B \in \mathscr{B} \cap A$, then

$$\varliminf_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}A) \geqq \frac{\mu(B)}{\mu(A)}$$

and

$$\varlimsup_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}A) = 1 - \varliminf_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu((A-B) \cap T^{-k}A)$$

$$\leqq 1 - \frac{\mu(A-B)}{\mu(A)} = \frac{\mu(B)}{\mu(A)}$$

i.e.

$$\text{(1.8)} \qquad \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}A) \xrightarrow[n\to\infty]{} \frac{\mu(B)}{\mu(A)} \qquad \forall B \in \mathscr{B} \cap A.$$

Now let $B, C \in \mathscr{B} \cap A$, then

$$\varliminf_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}C) \geqq \frac{\mu(B)\mu(C)}{\mu(A)^2}$$

and

$$\varlimsup_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}C)$$

$$= \varlimsup_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}A) - \varliminf_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}(A-C))$$

$$= \frac{\mu(B)}{\mu(A)} - \varliminf_{n\to\infty} \frac{1}{a_n(A)} \sum_{k=0}^{n} \mu(B \cap T^{-k}(A-C)) \text{ by (1.8)}$$

$$\leqq \frac{\mu(B)}{\mu(A)} - \frac{\mu(B)\mu(A-C)}{\mu(A)^2} = \frac{\mu(B)\mu(C)}{\mu(A)^2}. \qquad \text{Q.E.D.}$$

We note that theorem 3.2 in [7] (Foguel and Lin), when restricted to e.m.p.t.s, is equivalent to

(1.9) $$A \in R(T) \text{ iff } A \text{ satisfies } (1.8).$$

If $A, B \in R(T)$ then a double application of (1.7) yields that

(1.10) $$\frac{a_n(B)}{a_n(A)} \xrightarrow[n\to\infty]{} \frac{\mu(B)^2}{\mu(A)^2}.$$

We are now in a position to show that T ergodic does not imply that $R(T) \neq \emptyset$.

EXAMPLE 1.2 — *an* i.e.m.p.t. T *with* $R(T) = \emptyset$.

In [9], Hajian, Ito and Kakutani constructed an i.e.m.p.t. $(X, \mathcal{B}, \mu, T)$ together with an invertible measurable transformation $Q: X \to X$ with the properties that (i) $QT = TQ$, (ii) $\mu Q = \alpha\mu$ $(\alpha \neq 1)$. Now, if $A \in R(T)$, then, by the invertibility of Q, $QA \in R(T)$ and

$$\alpha = \frac{\sum_{k=1}^{n} \mu(Q(A \cap T^{-k}A)}{\sum_{k=1}^{n} \mu(A \cap T^{-k}A)} = \frac{a_n(QA)}{a_n(A)} \xrightarrow[n\to\infty]{} \frac{\mu(QA)^2}{\mu(A)^2} = \alpha^2 \text{ by (1.10).}$$

This contradicts $\alpha \neq 1$ and so $R(T) = \emptyset$.

We will say that T *is weakly rationally ergodic* (*w.r.e.*) iff T is ergodic and $R(T) \neq \emptyset$. Now, if T is w.r.e. then, by (1.10), there are sequences $\{a_n(T)\}_{n=1}^{\infty}$ such that

(1.11) $$\frac{a_n(A, T)}{a_n(T)} \xrightarrow[n\to\infty]{} \mu(A)^2 \qquad \forall A \in R(T).$$

We will call any sequence $\{a_n(T)\}$ satisfying (1.10) a *return sequence of* T (return sequence, because $a_n(A, T)$ measures the expected number of times points of A return to A under T before time n when $\mu(A) = 1$).

We denote by $\mathcal{A}(T)$ the class of sequences $\{\{b_n\}_{n=1}^{\infty}: b_n/a_n(T) \longrightarrow_{n\to\infty} c$ for some $c \in (0, \infty)$, and some return sequence $\{a_n(T)\}\}$.

The object $\mathcal{A}(T)$ will be called the *asymptotic type of* T. We reserve the right to abuse our notation in the following way: T will be said to be *of asymptotic type* $\{f(n)\}_n$ if $f(n)/a_n(T) \to c \in (0, \infty)$ for some return sequence $\{a_n(T)\}$, and this will be written: $\mathcal{A}(T) = \{f(n)\}$ (e.g. if $\mu(X) < \infty$ then by (1.1): $\mathcal{A}(T) = \{n\}$).

The property (1.5) could be viewed as a "weak L^1 ergodic theorem on $A \in \mathcal{F}$" since by (1.9): $A \in R(T)$ iff

$$\frac{1}{a_n(A)} \sum_{k=1}^{n} \chi_A \circ T^k \xrightarrow[n\to\infty]{} \frac{1}{\mu(A)} \text{ weakly in } L^1(A).$$

Analogously, "strong L^p ergodic theorems" could be considered. We will study these in a future publication. Here, we consider a condition that would be implied by a "strong L^2 ergodic theorem on A":

$$\sup_{n\geqq 1}\int_A \left(\frac{1}{a_n(A)}\sum_{k=1}^{n}\chi_A\circ T^k\right)^2 d\mu<\infty. \tag{1.12}$$

If T is ergodic and there is an $A\in\mathcal{F}$ satisfying (1.12), we will say that T is *rationally ergodic* (*r.e.*); and the collection of sets A satisfying (1.12) will be denoted by $B(T)$. To justify the choice of name we show that rational ergodicity is indeed stronger than weak rational ergodicity.

First, the notion of an *induced transformation* (Kakutani [12]) is recalled. Let T be a conservative m.p.t. and let $A\in\mathcal{F}$. For

$$x\in A\cap T^{-n}A-\bigcup_{k=1}^{n-1}T^{-k}A \quad (\text{where } n\geqq 1): \tag{1.13}$$

Let $T_Ax=T^nx$ then ([12]) T_A: $A\to A$ and $(A,\mathcal{B}\cap A,\mu_A,T_A)$ is a m.p.t., and an i.e.m.p.t. if T is an i.e.m.p.t.

LEMMA 1.3. *Let T be an i.e.m.p.t. and let $A\in\mathcal{F}$, then $\forall B,C\in\mathcal{B}\cap A$ and $n\geqq 1$*

$$\left|\sum_{k=0}^{n}\mu(T_AB\cap T^{-k}C)-\sum_{k=0}^{n}\mu(B\cap T^{-k}C)\right|\leqq\mu(A). \tag{1.14}$$

PROOF. We prove the lemma for T^{-1}. Noting that $\chi_B(T^kx)=0\ \forall B\subseteq A$ whenever $T^kx\neq T_A^jx$ for all $j\geqq 1$, we see that for every $x\in A$, and $n\geqq 1$, $\exists k_n(x)$ such that

$$\sum_{k=0}^{n}\chi_B(T^kx)=\sum_{j=0}^{k_n(x)}\chi_B(T_A^jx) \quad \text{for every } x\in A,\ B\in\mathcal{B}\cap A. \tag{1.15}$$

Hence for every $x\in A$, $B\in\mathcal{B}\cap A$

$$\begin{aligned}\sum_{k=0}^{n}\chi_{T_A^{-1}B}(T^kx)&=\sum_{j=0}^{k_n(x)}\chi_{T_A^{-1}B}(T_A^jx)\\ &=\sum_{j=1}^{k_n(x)+1}\chi_B(T_A^jx)\\ &=\sum_{j=0}^{k_n(x)}\chi_B(T_A^jx)+\chi_B(T_A^{k_n(x)+1}x)-\chi_B(x)\\ &=\sum_{k=0}^{n}\chi_B(T^kx)+\chi_B(T_A^{k_n(x)+1}x)-\chi_B(x).\end{aligned}$$

Thus

$$(1.16)\quad \left|\sum_{k=0}^{n}\chi_{T_A^{-1}B}(T^kx)-\sum_{k=0}^{n}\chi_B(T^kx)\right|\leqq 1 \text{ for every } B\in\mathcal{B}\cap A,\ x\in A.$$

Integrating (1.16) on $C\in\mathcal{B}\cap A$, we obtain

$$(1.17)\quad \left|\sum_{k=0}^{n}\mu(T^{-k}T_A^{-1}B\cap C)-\sum_{k=0}^{n}\mu(T^{-k}B\cap C)\right|\leqq\mu(C)$$

$$\leqq\mu(A) \text{ for } B,C\in\mathcal{B}\cap A,\ n\geqq 1.$$

Now, (1.17) and the assumption that T is an i.e.m.p.t. yield

$$(1.18)\quad \left|\sum_{k=0}^{n}\mu(T_A^{-1}B\cap T^kC)-\sum_{k=0}^{n}\mu(B\cap T^kC)\right|$$

$$\leqq\mu(A) \text{ for every } B,C\in\mathcal{B}\cap A,\ n\geqq 1.$$

This is (1.14) for T^{-1}. Q.E.D.

THEOREM 1.4. *Let T be an i.e.m.p.t. If T is r.e. then T is w.r.e.*

PROOF. We prove that $B(T)\subseteq R(T)$. Let $A\in B(T)$, and $\phi_n=(1/a_n(A))\Sigma_{k=0}^{n}\chi_A\circ T^k$; then

$$\phi_n\in L^2(A)\qquad n\geqq 1$$

and

$$(1.19)\qquad \|\phi_n\|_2\leqq M\qquad n\geqq 1.$$

It is a well known property of Hilbert spaces that (1.19) is sufficient for every subsequence of $\{\phi_n\}$ to have a subsequence weakly convergent in $L^2(A)$ (a Hilbert space).

Now, if $\phi_{n_k}\to\phi$ weakly in $L^2(A)$, then by (1.14) $\phi\circ T_A=\phi$ a.e., and hence, by the ergodicity of T_A and the fact that $\int_A\phi_n d\mu=1\forall n$, $\phi=1/\mu(A)$ a.e. This means that every subsequence of $\{\phi_n\}$ has a subsequence converging to $1/\mu(A)$ in $L^2(A)$—i.e. $\phi_n\to 1/\mu(A)$ weakly in $L^2(A)$ as $n\to\infty$. In particular

$$(1.20)\qquad \frac{1}{a_n(A)}\sum_{k=0}^{n}\mu(B\cap T^{-k}A)\xrightarrow[n\to\infty]{}\frac{\mu(B)}{\mu(A)}\quad\forall B\in\mathcal{B}\cap A.$$

The same argument applies to T^{-1}, so, since T is an m.p.t.,

$$(1.21)\qquad \frac{1}{a_n(A)}\sum_{k=0}^{n}\mu(A\cap T^{-k}B)\xrightarrow[n\to\infty]{}\frac{\mu(B)}{\mu(A)}\qquad\forall B\in\mathcal{B}\cap A.$$

Now choose any $C \in \mathcal{B} \cap A$ and let $\psi_n = (1/a_n(A))\Sigma_{k=0}^{n}\chi_C \circ T^k$, then $\|\psi_n\|_2 \leqq \|\phi_n\|_2 \leqq M$, $n \geqq 1$ and an argument similar to that leading to (1.20) (combined with (1.21)) shows that

$$\psi_n \xrightarrow[n\to\infty]{} \frac{\mu(C)}{\mu(A)^2} \quad \text{weakly in } L^2(A).$$

In particular

$$\frac{1}{a_n(A)}\sum_{k=0}^{n} \mu(B \cap T^{-k}C) \xrightarrow[n\to\infty]{} \frac{\mu(B)\mu(C)}{\mu(A)^2} \qquad \forall B, C \in \mathcal{B} \cap A.$$

Q.E.D.

Attention will henceforth be confined to rationally ergodic m.p.t.s (r.e.m.p.t.s) since the author knows of no w.r.e.m.p.t. that is not r.e.

Advantages of r.e. over w.r.e. will become evident in the next section, where we will define some metric relations between m.p.t.s, and show that the asymptotic type of r.e.m.p.t.s is invariant for all of them.

§2. Isomorphisms and other metric relations

Let $(X, \mathcal{B}, \mu, T)$ and $(X', \mathcal{B}', \mu', T')$ be m.p.t.s. Let $0 < c < \infty$. We will say that *π is a c-map of T onto T'* (π: $T \xrightarrow{c} T'$) iff π: $X \to X'$ is a map (defined μ-a.e.) s.t. $\pi^{-1}\mathcal{B}' \subseteq \mathcal{B}$, $\mu \circ \pi^{-1} = c\mu'$ and $\pi T = T'\pi$. If, in addition, π is invertible (i.e. π is one to one where defined and $\pi^{-1}\mathcal{B}' = \mathcal{B}$), then we will say that *π is an invertible c-map of T onto T'* (π: $T \xleftrightarrow{c} T'$). (Note that if π: $T \xleftrightarrow{c} T'$ then π^{-1}: $T' \xleftrightarrow{c^{-1}} T$.)

We say that *T' is a c-factor of T* ($T \xrightarrow{c} T'$) iff there is a c-map of T onto T'; and that *T' is a factor of T* ($T \to T'$) iff $T \xrightarrow{c} T'$ for some $c \in (0, \infty)$.

It is necessary to introduce the constant c because the measure spaces are not normalised. If $\mu(X)$, $\mu'(X') < \infty$ and $T \xrightarrow{c} T'$ then $c = \mu(X)/\mu(X')$. When $\mu(X)$, $\mu'(X') = \infty$, there is no such *a priori* restriction on the values of c for which $T \xrightarrow{c} T'$.

We say that *T is similar to T'* ($T \sim T'$) iff T and T' are both factors of the same m.p.t.

All transformations preserving finite measures are pairwise similar since any two of them are both factors of their Cartesian product. It is comparatively rare that transformations preserving ∞ measure are similar. We do not know if similarity is an equivalence relation.

If $T \to T'$ and $T' \to T$ then we say that *T is weakly isomorphic with T'* ($T \approx T'$).

If there is an invertible c-map of T onto T' for some $c \in (0,\infty)$ then we say that *T is isomorphic with T'*$(T \leftrightarrow T')$. Clearly

$$T \leftrightarrow T' \Rightarrow T \approx T' \Rightarrow T \to T' \Rightarrow T \sim T'. \tag{2.1}$$

Now let T be an e.m.p.t. and T' be a r.e.m.p.t. Assume $\pi\colon T \xrightarrow{c} T'$, then

$$\pi^{-1}B(T') \subseteq B(T) \textit{ and, hence, } T \textit{ is r.e.} \tag{2.2}$$

Moreover, let $A \in B(T')$; then

$$\frac{a_n(T')}{a_n(T)} \sim \frac{a_n(A,T')}{\mu'(A)^2 a_n(T)} = \frac{a_n(\pi^{-1}A,T)}{c\mu'(A)^2 a_n(T)} \xrightarrow[n\to\infty]{} \frac{\mu(\pi^{-1}A)^2}{c\mu'(A)^2} = c \tag{2.3}$$

by (1.11), and since

$$a_n(\pi^{-1}A,T) = \sum_{k=1}^{n} \mu(\pi^{-1}(A \cap T'^{-k}A)) = ca_n(A,T').$$

Now from (2.3) it follows that

$$\mathscr{A}(T) = \mathscr{A}(T') \tag{2.4}$$

and

$$\begin{gathered} \textit{If } T \textit{ is an e.m.p.t., } T' \textit{ a r.e.m.p.t. and } T \to T' \textit{ then} \\ \exists\,!\; c \in (0,\infty) \textit{ s.t. } T \xrightarrow{c} T'. \end{gathered} \tag{2.5}$$

We note, in order to obtain analagous results for T' w.r.e., we would either have to assume that $\pi\colon T \xleftrightarrow{c} T'$ or that T is also w.r.e.

Moreover, we do not know if the asymptotic types of similar w.r.e.m.p.t.s coincide.

The rest of this section is devoted to showing that two similar r.e.m.p.t.s do have the same asymptotic type.

The next three results are technical, and show that (for the purpose of calculating asymptotic type) two similar r.e.m.p.t.s can be considered as factors of one e.m.p.t.

Let $(X, \mathscr{B}, \mu, T)$ be an invertible m.p.t. By an *ergodic decomposition of T* is meant a probability space (Ω, Σ, P) and a collection of measures $\{\mu_\omega\}_{\omega\in\Omega}$ such that for every countable ring $\mathscr{R} \subseteq \mathscr{F}$ $\exists \Omega_{\mathscr{R}} \in \Sigma$ s.t. $P(\Omega_{\mathscr{R}}) = 1$ and

$$\mu_\omega \text{ is a measure on } (X, \mathscr{R})\ \forall \omega \in \Omega_{\mathscr{R}}, \tag{2.6}$$

$$(X, \mathcal{B}_\omega, \mu_\omega, T)\text{—denoted by } T_\omega\text{—is an e.m.p.t. (where } \mathcal{B}_\omega \text{ is the } \mu_\omega\text{-completion of } \mathcal{R}), \tag{2.7}$$

$$\forall A \in \mathcal{R}: \mu_\omega(A) \text{ is a measurable function of } \omega \text{ and } \int_\Omega \mu_\omega(A)dP(\omega) = \mu(A), \tag{2.8}$$

and

$$\mu_\omega(X) > 0 \quad \forall \omega \in \Omega_{\mathcal{R}}. \tag{2.9}$$

We will denote the above ergodic decomposition of T by $(\Omega, \Sigma, P, \{\mu_\omega\})$.

THEOREM 2.1 [22]. *If $(X, \mathcal{B}, \mu, T)$ is a m.p.t.; $(X, \mathcal{B}, \mu)$ is a Lebesgue space i.e. is generated by a countable ring $\mathcal{R}$ which separates points (see [21]) and $\mu(X) = 1$: then there is an ergodic decomposition of T*

$$(\Omega, \Sigma, P, \{\mu_\omega\}) \text{ such that } \mu_\omega(x) = 1 \quad \forall \omega \in \Omega.$$

The following seems to be well known:

PROPOSITION 2.2. *Let $(X, \mathcal{B}, \mu, T)$ be an m.p.t., $\mu(X) = 1$ and let $\mathcal{C} \subseteq \mathcal{B}$ be a countably generated, T-invariant σ-algebra such that $(X, \mathcal{C}, \mu, T)$ is ergodic. If $(\Omega, \Sigma, P, \{\mu_\omega\})$ is an ergodic decomposition of T s.t. $\mu_\omega(X) = 1$ $\forall \omega$, then $\exists \Omega' \subseteq \Omega$ s.t. $P(\Omega') = 1$ and s.t.*

$$\mu_\omega(C) = \mu(C) \forall \omega \in \Omega', \ C \in \mathcal{C}. \tag{2.10}$$

LEMMA 2.3. *Let $(X, \mathcal{B}, \mu, T)$ be an invertible m.p.t. $(\mu(X) \leqq \infty)$ and let $(X, \mathcal{B}, \mu)$ be a Lebesgue space. Let $\mathcal{C} \subseteq \mathcal{B}$ be a σ-finite, countably generated T-invariant σ-algebra s.t. $(X, \mathcal{C}, \mu, T)$ is ergodic then:*

(i) $\exists$ *an ergodic decomposition for T.*

(ii) *If $(\Omega, \Sigma, P, \{\mu_\omega\})$ is an ergodic decomposition for T then $\exists$ $c : \Omega \to (0, \infty)$ measurable, $\Omega' \subseteq \Omega$ s.t. $P(\Omega') = 1$ such that*

$$\forall C \in \mathcal{C}, \ \omega \in \Omega': \mu_\omega(C) = c(\omega)\mu(C). \tag{2.11}$$

PROOF. (i) is theorem 6.1 of [18].

(ii) Choose $C \in \mathcal{C}$, then $(\Omega, \Sigma, P, \{\mu_\omega\})$ is an ergodic decomposition for $(C, \mathcal{B} \cap C, \mu_C, T_C)$ and $c(\omega) = \mu_\omega(C)$ is a measurable function from Ω into $(0, \infty)$. Let $d\bar{P} = c dP/\mu(C)$ then $(\Omega, \Sigma, \bar{P}, \{(1/c(\omega))\mu_\omega\})$ is an ergodic composition for $(C, \mathcal{B} \cap C, \mu_C, T_C)$ s.t. $\mu_\omega(C)/c(\omega) = 1$ $\forall \omega$. Hence by Proposition 2.2:

$$\exists \Omega' \in \Sigma,\ P(\Omega') = 1 \text{ s.t. } \frac{\mu_\omega(A)}{c(\omega)} = \frac{\mu(A)}{\mu(C)} \qquad \forall A \in \mathscr{C} \cap C, \qquad \omega \in \Omega'$$

$$\text{i.e. } \mu_\omega(A) = \frac{c(\omega)}{\mu(C)}\mu(A) \quad \forall A \in \mathscr{C} \cap C,\ \omega \in \Omega'. \tag{2.12}$$

Now, as $C \in \mathscr{C}$: $\bigcup_{n=1}^{\infty} T^{-n}C = X \bmod \mu$ and so (2.12) extends under T-iteration to $\mathscr{C}$.

THEOREM 2.4. *Let T_1 and T_2 be similar r.e.m.p.t.s, then*

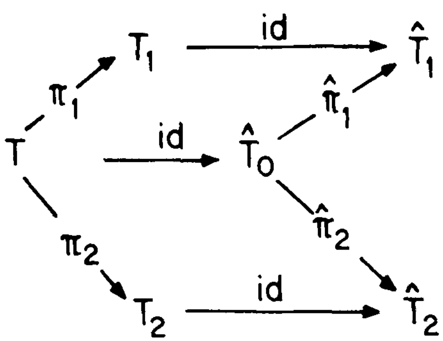
$$\mathscr{A}(T_1) = \mathscr{A}(T_2).$$

PROOF. Let T be an invertible m.p.t. and assume π_1: $T \to T_1$ and π_2: $T \to T_2$. For $i = 1,2$ choose $A_i \in B(T_i)$ and let $\alpha_i = \{A_i, A_i^c\}$. Let $\hat{X}_i = \{\{\chi_{A_i} \circ T^n(x)$: $n \in \mathbf{Z}\}\ x \in X_i\}$, $\hat{\mathscr{B}}_i$ = the σ-algebra generated by $\bigvee_{n=-\infty}^{\infty} T_i^n \alpha_i$ (Note that sets in $\hat{\mathscr{B}}_i$ are subsets of $\hat{X}_i$), $\hat{\mu}_i = \mu_i \mid_{\hat{\mathscr{B}}_i}$ and $\hat{T}_i$ = the shift on $\hat{X}_i$.

Furthermore, let $\alpha_0 = \pi_2^{-1}\alpha_1 \vee \pi_2^{-1}\alpha_2 = \{\bar{A}_1 - \bar{A}_2,\ \bar{A}_1 \cap \bar{A}_2,\ \bar{A}_2 - \bar{A}_1,\ (\bar{A}_1 \cup \bar{A}_2)^c\} = \{B_1, B_2, B_3, B_4\}$ where $\bar{A}_i = \pi_i^{-1}A_i$ $(i = 1,2)$.

Let $\hat{X}_0 = \{\{\sum_{i=1}^{4} i\chi_{B_i} \circ T^n(x);\ n \in \mathbf{Z}\}: x \in X\}$, $\hat{\mathscr{B}}_0$ = the σ-algebra generated by $\bigvee_{n=-\infty}^{\infty} T^n\alpha_0$, $\hat{\mu}_0 = \mu \mid_{\hat{\mathscr{B}}_0}$ and $\hat{T}_0$ the shift on $\hat{X}_0$.

Then, for $i = 0,1,2$ $(\hat{X}_i, \hat{\mathscr{B}}_i, \hat{\mu}_i)$ are Lebesgue spaces, and the following diagram represents the relationship between the above transformations:



Moreover, $\hat{T}_1$ and $\hat{T}_2$ are r.e.m.p.t.s. Thus, by (2.5)

$$\mathscr{A}(T_i) = \mathscr{A}(\hat{T}_i) \qquad (i = 1,2). \tag{2.13}$$

Now, by Lemma 2.3 (i) $\exists$ an ergodic decomposition $(\Omega, \Sigma, P, \{\mu_\omega\})$ for $\hat{T}_0$. Now, $\hat{\pi}_i^{-1}\hat{\mathscr{B}}_i$ $(i = 1,2)$ are countably generated, T_0-invariant, σ-finite sub-σ-algebras of $\hat{B}_0$ and $(\hat{X}_0,\ \pi_i^{-1}\hat{B}_i,\ \hat{\mu}_0,\ \hat{T}_0)$ $(i = 1,2)$ are ergodic. Hence by Lemma 2.3 (ii) $\exists \Omega' \subseteq \Omega$ s.t. $P(\Omega') = 1$, c_1, c_2: $\Omega' \to (0,\infty)$ s.t. $\forall \omega \in \Omega'$

$$\mu_\omega(\hat{\pi}_i^{-1}A_i) = c_i(\omega)\mu_i(A) \quad \forall A \in \hat{\mathscr{B}}_i \quad (i = 1,2).$$

But this just means that for $\omega \in \Omega'$, $\hat{T}_\omega \to \hat{T}_1$ and $\hat{T}_\omega \to \hat{T}_2$. So we have shown that $\hat{T}_1$ and $\hat{T}_2$ are factors of an e.m.p.t. Thus, by (2.4) and (2.13)

$$\mathcal{A}(T_1) = \mathcal{A}(\hat{T}_1) = \mathcal{A}(\hat{T}_\omega) = \mathcal{A}(\hat{T}_2) = \mathcal{A}(T_2) \qquad \text{Q.E.D.}$$

§3. Markov shifts and recurrent events

First, we recall briefly the definition of a Markov shift (see Chung [4] and Harris and Robbins [11]).

Let S be a countable set, the *state space*, and $P = \{p_{s,t}\}_{s,t \in S}$ be a stochastic matrix [4] (sometimes called transition matrix).

If P has a stationary distribution $m = \{m_s(P)\}_{s \in S}$ (satisfying $m_s \geqq 0$, $\Sigma_{s \in S} m_s p_{s,t} = m_t \ \forall t \in S$) then we can define the (two-sided) Markov shift of P, (P, m), as follows:

$$X = S^{\mathbf{Z}} = \{(\cdots x_{-1}, x_0, x_1 \cdots) : x_n \in S \ \ \forall n \in \mathbf{Z}\}$$

$\mathcal{B}$ is the σ-algebra generated by cylinder sets; μ_P is the σ-finite measure generated by

$$\mu_P([x_n = s_n, x_{n+1} = s_{n+1} \cdots x_{n+k} = s_{n+k}])$$

$$= m_{s_n} p_{s_n, s_{n+1}} \cdots p_{s_{n+k-1}, s_{n+k}} \quad \forall n \in \mathbf{Z},\ k \geqq 1,\ s_n \cdots s_{n+k} \in S$$

where $[x_n = s_n, \cdots, x_{n+k} = s_{n+k}]$ denotes the set $\{x \in X : x_n = s_n \cdots x_{n+k} = s_{n+k}\}$. $(X, \mathcal{B}, \mu_P, T_P)$ is an invertible m.p.t. and is known as the *(two-sided) Markov shift with transition matrix P* (and stationary distribution m).

Because of

THEOREM 3.1 [11]. *T_P is ergodic iff P is irreducible recurrent.*

and

THEOREM 3.2 [4]. *If P is irreducible, recurrent then there is a stationary distribution for P, unique up to multiplication by a constant.*

it is evident that the measure space, upon which an ergodic Markov shift is defined, is unique up to constant multiplication of the measure.

Let $m(P)$ be a stationary distribution of the (irreducible, recurrent) stochastic matrix P. Then P is called *positive* or *null* according to whether $(\Sigma_{s \in S} m_s(P))^{-1}$ is positive or zero respectively. There are many irreducible, null recurrent stochastic matrices, and their Markov shifts are i.e.m.p.t.s of infinite measure spaces. We will show in this section that any i.e.m.p.t., having one as a factor, is rationally ergodic.

It is convenient to introduce an example at this stage. Let $\{f_n\}_{n=1}^{\infty}$ be s.t.

$$(3.1) \qquad f_n \geqq 0, \qquad \Sigma_{n=1}^{\infty} f_n = 1.$$

Define

$$(3.2) \qquad f(\lambda) = \sum_{n=1}^{\infty} f_n \lambda^n, \quad u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n = \frac{1}{1 - f(\lambda)} \quad \underline{u} = \{u_n\}_{n=0}^{\infty}.$$

The sequence $\underline{u}$ is called *a recurrent renewal sequence*, as is any sequence obtained in this way from an $\{f_n\}$ satisfying (3.1). Conversely, any recurrent renewal sequence $\underline{u}$ has a unique probability distribution $\{f_n(\underline{u})\}_n$ on $\mathbf{N}$ associated with it. Kaluza, in [14], showed that *if* $\underline{u} = \{u_n\}_{n=0}^{\infty}$ *satisfies* $u_0 = 1$, $\Sigma_{n=0}^{\infty} u_n = \infty$, *&* $u_{n+1}/u_n \uparrow 1$ *as* $n \uparrow \infty$ *then* $\underline{u}$ *is a recurrent renewal sequence.* This theorem identifies many recurrent renewal sequences (see Kingman [16] for an exposition).

Let $\underline{u}$ be a recurrent renewal sequence, define a stochastic matrix as follows:

$$p_{s,t} = \begin{cases} f_t(\underline{u}) & \text{if } s = 1 \\ 1 & \text{if } s \geqq 2 \quad t = s - 1, \\ 0 & \text{otherwise} \end{cases} \quad P_{\underline{u}} = \{p_{s,t}\}_{s,t \in \mathbf{N}} \quad (S = \mathbf{N}).$$

Then ([4]) $p_{11}^{(n)} = u_n$, $n \geqq 0$; and $P_{\underline{u}}$ is irreducible recurrent, and has the stationary distribution $m(\underline{u})$ given by

$$m_s(\underline{u}) = \sum_{t=s}^{\infty} f_t(\underline{u}) \quad (s \geqq 1).$$

It is seen that $P_{\underline{u}}$ is positive or null according to whether $(\Sigma_{n=1}^{\infty} n f_n(\underline{u}))^{-1}$ is positive or zero respectively.

We denote by $T_{\underline{u}}$ the Markov shift with transition matrix $P_{\underline{u}}$, and call it the *Markov shift of the (recurrent) renewal sequence* $\underline{u}$.

We now examine the more general property of i.e.m.p.t.s, of having some Markov shift as factor.

Let T be an e.m.p.t. and let $A \in \mathcal{F}$. We will say that A is a *recurrent event* iff $\forall 0 \leqq n_1, \leqq \cdots \leqq n_k$

$$\mu_A\left(\bigcup_{j=1}^{k} T^{-n_j} A\right) = \prod_{j=1}^{k} \mu_A(T^{-(n_j - n_{j-1})} A)$$

where $n_0 = 0$ and $\mu_A(B) = \mu(A \cap B)/\mu(B)$. (Compare this to the definition on p. 307 of [5].) We denote by $M(T)$ the collection of recurrent events for T. We shall say that T *admits recurrent events* iff $M(T) \neq \varnothing$.

It follows from the renewal theorem [4] that any e.m.p.t. admitting recurrent events is of zero-type [8].

Now, if T_P is some ergodic Markov shift and $s \in S$ then $[x_0 = s] \in M(T_P)$. Conversely, it is not hard to prove that if $A \in M(T)$ then $\underline{u}(A) = \{\mu_A(T^{-n}A)\}_{n=0}^{\infty}$ is a renewal sequence and $T \to T_{\underline{u}(A)}$. We sum this up in the following proposition.

PROPOSITION 3.3. *An e.m.p.t. T admits recurrent events iff it has some Markov shift as a factor.*

Thus, we see that the M in $M(T)$ is in honour of Markov.

THEOREM 3.4. *Let T be an i.e.m.p.t. If T admits recurrent events then T is rationally ergodic. Moreover,* $M(T) \subseteq B(T)$.

PROOF. It is sufficient to show that $M(T) \subseteq B(T)$. To this end, let $A \in M(T)$ and $u_n(A) = u_n = \mu_A(T^{-n}A)$. Then

$$\begin{aligned}
\int_A \left(\sum_{k=0}^{n} \chi_A \circ T^k\right)^2 d\mu &= \sum_{k=0}^{n}\sum_{l=0}^{n} \mu(A \cap T^{-k}A \cap T^{-l}A) \\
&\leqq 2\sum_{k=0}^{n}\sum_{l=k}^{n} \mu(A \cap T^{-k}A \cap T^{-l}A) \\
&= 2\mu(A)\sum_{k=0}^{n}\sum_{l=k}^{n} u_k u_{l-k} \\
&= 2\mu(A)\sum_{k=0}^{n}\sum_{l=0}^{n-k} u_k u_l \\
&\leqq 2\mu(A)\left(\sum_{k=0}^{n} u_k\right)^2 = \frac{2}{\mu(A)} a_n(A)^2
\end{aligned}$$

i.e. $A \in B(T)$. Q.E.D.

We note that example 3.2 of [7] is equivalent to the result: *ergodic one-sided Markov shifts with transition probabilities satisfying the strong ratio limit property are weakly rationally ergodic.*

Having shown that ergodic Markov shifts are rationally ergodic, we now calculate the asymptotic type of an ergodic Markov shift in terms of its transition matrix.

Let T_P be the ergodic Markov shift with transition matrix P and considered with stationary distribution $m(P)$. Let $s \in S$, then, by Theorem 3.4, $[x_0 = s] \in B(T_P)$, and so

$$a_n(T_P) \sim \frac{a_n([x_0 = s], T_P)}{m_s^2} = \frac{1}{m_s} \sum_{k=0}^{n} p_{s,s}^{(k)} \tag{3.1}$$

for any return sequence $a_n(T_P)$

and

$$\mathcal{A}(T_P) = \left\{ \sum_{k=0}^{n} p_{s,s}^{(k)} \right\}. \tag{3.2}$$

Applying (3.1) and (3.2) to (2.3) and Theorem 2.4, respectively, we obtain the following:

THEOREM 3.5. *Let T_P and T_Q be ergodic Markov shifts with transition matrices P and Q respectively. Let s and t be states of P and Q, then*

(i)

$$T_P \xrightarrow{c} T_Q \Rightarrow \frac{\sum_{k=0}^{n} q_{t,t}^{(k)}}{\sum_{k=0}^{n} p_{s,s}^{(k)}} \xrightarrow[n\to\infty]{} \frac{m_t(Q)}{m_s(P)} \cdot c, \tag{3.3}$$

(ii)

$$T_P \sim T_Q \Rightarrow \exists \lim_{n\to\infty} \frac{\sum_{k=0}^{n} q_{t,t}^{(k)}}{\sum_{k=0}^{n} p_{s,s}^{(k)}} \in (0, \infty). \tag{3.4}$$

We note that the lemma on p. 204 of [23] means that: *If ω is a non-increasing renewal sequence then*

$$\mathcal{A}(T_P) = \mathcal{A}(T_Q) \Rightarrow T_P \times T_\omega \text{ and } T_Q \times T_\omega \text{ are simultaneously conservative or dissipative (where } T_P \text{ and } T_Q \text{ are ergodic Markov shifts).} \tag{3.5}$$

This means that $\mathcal{A}(\cdot)$ refines the invariants of [23] when restricted to ergodic Markov shifts. (The invariants of [23] are actually invariants for similarity.)

In [18] and [23] uncountable collections of non-isomorphic ergodic Markov shifts were constructed. We point out that, since the shifts in these collections can be separated by the invariants of [23], they in fact have distinct asymptotic types.

We now show that there is a connection between the asymptotic type, and something like the ergodic index (cf. [13]) of a transformation admitting recurrent events.

Let T be an i.e.m.p.t. and T_n its n-fold Cartesian product. Define (as in [13]) the *ergodic index of* T to be:

$$e(T) = \max\{n \geqq 1:\ T_n \text{ is ergodic}\};$$

the *index of conservativity of* T to be:

$$c(T) = \max\{n \geqq 1:\ T_n \text{ is conservative}\};$$

and the *dissipating index of* T to be:

$$d(T) = \min\{n \geqq 1:\ T_n \text{ is dissipative}\}$$

where $e(T) = \infty$ ($c(T)$, $d(T) = \infty$) means that T_n is ergodic (conservative — not dissipative) for every $n \geqq 1$.

It is shown in [20] that:

$$e(T) = c(T) = d(T) - 1 \textit{ if } T \textit{ is an aperiodic Markov shift}$$

and that

$$c(T) = d(T) - 1 \textit{ if } T \textit{ is an ergodic Markov shift.}$$

It follows from this that

$$c(T) = d(T) - 1 \textit{ if } T \textit{ admits recurrent events.} \tag{3.6}$$

The following shows a connection between asymptotic type and index of conservativity:

PROPOSITION 3.6. *If T is an e.m.p.t. admitting recurrent events and $a_n(T)$ is a return sequence for T then*

$$\overline{\lim_{n\to\infty}} \frac{\log a_n(T)}{\log n} \leqq 1 - \frac{1}{c(T)+1} \tag{3.7}$$

(*and hence* $\overline{\lim}_{n\to\infty} \log a_n(T)/\log n = 1 \Rightarrow c(T) = \infty$).

PROOF. Let $A \in M(T)$, then it follows from Theorems 3.1 and 3.2, and (3.6) that if $c(T) < \infty$,

$$\sum_{n=0}^{\infty} u_n(A)^{c(T)+1} < \infty. \tag{3.8}$$

Thus it is sufficient to prove the following: If $u_n \geqq 0$ and $\sum_{n=1}^{\infty} u_n^{\beta} < \infty$ (where $\beta > 1$) then

$$\varlimsup_{n\to\infty}\frac{\log\sum_{k=1}^{n}u_k}{\log n}\leqq 1-\frac{1}{\beta}. \tag{3.9}$$

By Hölder's inequality

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}u_n &\leqq \left(\sum_{n=1}^{\infty}u_n^\beta\right)^{1/\beta}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\alpha\beta/(\beta-1)}\right)^{1-1/\beta}\\ &= M_\alpha<\infty \quad \forall\alpha>1-\frac{1}{\beta}.\end{aligned} \tag{3.10}$$

Hence, $\forall\alpha>1-1/\beta$ and $n\geqq 1$

$$\frac{1}{n^\alpha}\sum_{k=1}^{n}u_k\leqq\sum_{k=1}^{n}\frac{1}{k^\alpha}u_k\leqq M_\alpha<\infty,\qquad \alpha>1-\frac{1}{\beta}$$

i.e.

$$\log\sum_{k=1}^{n}u_k\leqq\log M_\alpha+\alpha\log n\qquad\text{where } M_\alpha<\infty,\qquad \forall\alpha>1-\frac{1}{\beta}. \tag{3.11}$$

From (3.11) follows

$$\varlimsup_{n\to\infty}\frac{\log\sum_{k=1}^{n}u_k}{\log n}\leqq\alpha,\qquad\forall\alpha>1-\frac{1}{\beta}$$

which is the same as (3.9). Q.E.D.

Theorem 1 of [23] has the same flavour as the above proposition.

§4. Entropy and normalised asymptotic type

In this section we will combine asymptotic type and entropy to obtain a stronger invariant for weak isomorphism of r.e.m.p.t.s with positive finite entropy. We call this invariant "normalised asymptotic type".

First we recall from [18] the definition of entropy for i.e.m.p.t.s preserving a σ-finite but not necessarily finite measure.

Let $(X,\mathcal{B},\mu,T)$ be an i.e.m.p.t. with $\mu(X)\leqq\infty$. In the case $\mu(X)<\infty$, we will let $h(T)$ denote the Kolmogorov–Sinai entropy of T w.r.t. the normalised measure $(1/\mu(X))\mu$ (see [3]). We shall need a theorem of Abramov ([1]) which states that: *if $\mu(X)<\infty$ and $A\in\mathcal{B}$ then*

$$\frac{\mu(A)}{\mu(X)}h(T_A)=h(T). \tag{4.1}$$

Now, let $\mu(X) \leqq \infty$. If $A, B \in \mathscr{F}$ and $A \subseteq B$ then since

$$T_A = (T_B)_A \tag{4.2}$$

we have by (4.1)

$$\frac{\mu(A)}{\mu(B)} h(T_A) = h(T_B). \tag{4.3}$$

Now, (4.3) combined with the fact that if $A, B \in \mathscr{F}$ then $A, B \subseteq A \cup B \in \mathscr{F}$ yields that

$$\exists \text{ a constant } \underline{h}(T) \text{ s.t. } \mu(A)h(T_A) = \underline{h}(T) \qquad \forall A \in \mathscr{F}. \tag{4.4}$$

We call this constant $\underline{h}(T)$ (as in [18]) *the entropy of* T.

In case $\mu(X) < \infty$

$$\underline{h}(T) = \mu(X)h(T). \tag{4.5}$$

Now let T, T' be i.e.m.p.t.s, $A \in \mathscr{F}'$ and assume $\pi: T \xrightarrow{c} T'$, then $\pi: T_{\pi^{-1}A} \to T'_A$ and so ([3]) $h(T_{\pi^{-1}A}) \geqq h(T'_A)$. Hence

$$\underline{h}(T) = \mu(\pi^{-1}A)h(T_{\pi^{-1}A}) \geqq c\mu'(A)h(T'_A) = c\underline{h}(T'). \tag{4.6}$$

This tells us that the following entropy classes are preserved under weak isomorphism: $\mathscr{E}_0 = \{T \text{ i.e.m.p.t.: } \underline{h}(T) = 0\}$, $\mathscr{E}_+ = \{T \text{ i.e.m.p.t.: } 0 < \underline{h}(T) < \infty\}$ and $\mathscr{E}_\infty = \{T \text{ i.e.m.p.t.: } \underline{h}(T) = \infty\}$. (We note that the T of Example 1.2 is not in $\mathscr{E}_+$.)

We now recall the calculation of the entropy of an ergodic Markov shift in terms of its transition matrix:

THEOREM 4.1 ([18]). *Let T_P be an ergodic Markov shift with transition matrix P and let $0 \in S$ be a state, then*

$$\underline{h}(T_P) = m_0(P)h_0(P), \tag{4.7}$$

where

$$h_0(P) = \sum_{s,t \in S} {}_0p^*_{0,s} P_{s,t} \log \frac{1}{p_{s,t}}.$$

PROOF. (Sketch) Let $\alpha_0 = \Sigma_{n=1}^\infty \{[x_0 = 0,\ x_1 = s_1 \cdots x_{n-1} = s_{n-1},\ x_n = 0],\ n \geqq 1,$ $s_1 \cdots s_{n-1} \in S - \{0\}\}$. Then ([12]) the $(T_P)_{[x_0=0]}$-iterates of α_0 are independent and generate $\mathscr{B} \cap [x_0 = 0]$. Hence $\underline{h}(T_P) = \mu_P([x_0 = 0])h((T_P)_{[x_0=0]}) = m_0(P)H(\alpha_0)$.

It is shown in [18] (by calculation) that $H(\alpha_0) = h_0(P)$. Q.E.D.

(We note that (by 4.7) if T_P is a Markov shift, then $\underline{h}(T_P) > 0$ automatically.)

If $\underline{u}$ is a recurrent renewal sequence, then Theorem 4.1 yields that

$$\underline{h}(T_{u}) = H(f(\underline{u})) = \sum_{n=1} f_n(\underline{u}) \log \frac{1}{f_n(\underline{u})}. \tag{4.8}$$

The following proposition gives many Markov shifts $T_P \in \mathscr{E}_+$.

PROPOSITION 4.2. *If $\underline{u}$ is a recurrent renewal sequence and $a_n = \Sigma_{k=0}^{n} u_k$ then $\Sigma_{n=1}^{\infty} 1/na_n < \infty \Rightarrow T_u \in \mathscr{E}_+$.*

PROOF. It is sufficient to show that $H(f(\underline{u})) < \infty$.

Step 1.

$$\sum_{n=1}^{\infty} (\log n) f_n < \infty \Rightarrow H(f) < \infty. \tag{4.9}$$

Proof of Step 1. Let $A = \{n \geqq 1: f_n \leqq 1/(n+1)^2\}$ then if $n \in A$, since the function $x \log(1/x)$ is increasing on $(0, \frac{1}{2})$

$$f_n \log \frac{1}{f_n} \leqq \frac{2 \log (n+1)}{(n+1)^2}. \tag{4.10}$$

If $n \notin A$, then

$$f_n \log \frac{1}{f_n} \leqq 2(\log (n+1)) f_n. \tag{4.11}$$

Using (4.10) and (4.11) we see that

$$\begin{aligned} H(f) &= \sum_{n \in A} f_n \log \frac{1}{f_n} + \sum_{n \notin A} f_n \log \frac{1}{f_n} \\ &\leqq \sum_{n=1}^{\infty} \frac{2 \log (n+1)}{(n+1)^2} + \sum_{n=1}^{\infty} 2(\log(n+1)) - f_n < \infty \end{aligned}$$

and Step 1 is taken.

Step 2.

$$\sum_{n=1}^{\infty} (\log n) f_n < \infty \quad \text{iff} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n} c_k < \infty \quad \textit{where} \quad c_k = \sum_{j=k+1}^{\infty} f_j.$$

Proof of Step 2.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} c_k = \sum_{k=1}^{\infty} c_k \sum_{n=k}^{\infty} \frac{1}{n^2} < \infty$$

iff $\Sigma_{n=1}^{\infty} c_n / n < \infty$ and

$$\sum_{n=1}^{\infty}\frac{c_n}{n}=\sum_{n=1}^{\infty}\frac{2}{n}\sum_{k=n+1}^{\infty}f_k=\sum_{k=2}^{\infty}f_k\sum_{n=1}^{k-1}\frac{1}{n}<\infty$$

iff $\Sigma_{n=1}^{\infty}(\log n)f_n<\infty$.

It is now sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{n}c_k<\infty \quad \text{iff} \quad \sum_{n=1}^{\infty}\frac{1}{na_n}<\infty.$$

To do this, we show

$$(n+1)\leqq\sum_{k=0}^{n}u_k\sum_{l=0}^{n}c_l\leqq 2(n+1). \tag{4.12}$$

Now, since $\Sigma_{k=0}^{\infty}u_k\lambda^k\,\Sigma_{l=0}^{\infty}c_l\lambda^l=1/(1-\lambda)\,\forall\lambda\in(0,1)$, we have

$$\sum_{k=0}^{n}u_kc_{n-k}=1 \qquad n\geqq 0. \tag{4.13}$$

Hence

$$\begin{aligned}(n+1)&=\sum_{m=0}^{n}\sum_{k=0}^{m}u_kc_{m-k}=\sum_{k=0}^{n}u_k\sum_{m=k}^{n}c_{m-k}=\sum_{k=0}^{n}u_k\sum_{l=0}^{n-k}c_l\\&\leqq\sum_{k=0}^{n}u_k\sum_{l=0}^{n}c_l\leqq(n+1)+\sum_{k=0}^{n}u_k\sum_{l=n-k}^{n}c_l\\&=(n+1)+\sum_{k=0}^{n}u_k\sum_{m=0}^{k}c_{n-m}\\&\leqq(n+1)+\sum_{k=0}^{n}(k+1)u_kc_{n-k}\quad\because\ c_n\downarrow\text{ as }n\uparrow\\&\leqq(n+1)+(n+1)\sum_{k=0}^{n}u_kc_{n-k}=2(n+1)\quad\text{by (4.13).}\end{aligned}$$

Q.E.D.

It also follows from Proposition 4.2 that if T (an i.e.m.p.t.) admits recurrent events and $\Sigma_{n=1}^{\infty}1/na_n(T)<\infty$ then

(i) *T is quasi-finite (in the sense of* [18]), and

(ii) *the entropy of the first return time partition of every recurrent event of T is finite.* (See [17] for a related result.)

We will now construct the advertised invariant for weak isomorphism of r.e.m.p.t.s in $\mathscr{E}_+$.

Let T be a r.e.m.p.t. $T\in\mathscr{E}_+$, if $\{a_n(T)\}$ is a return sequence of T then we let

$$\hat{a}_n(T) = \underline{h}\,(T) a_n(T) \tag{4.14}$$

and call the sequence $\{\hat{a}_n(T)\}$ (and any sequence asymptotically equal to it) a *normalised return sequence of T*.

We denote by $\mathcal{H}(T)$ the collection $\{\{a_n\}$ a sequence: $a_n/\hat{a}_n(T) \to 1$ as $n \to \infty$ for a normalised return sequence of $T\}$ (i.e. $\mathcal{H}(T)$ denotes the collection of normalised return sequences of T). We call $\mathcal{H}(T)$ the *normalised asymptotic type of T*, and prepare to abuse our notation in the following way:

we say that T is of *normalised asymptotic type* $\{f(n)\}$ if $f(n)/\hat{a}_n(T) \to 1$ for any normalised return sequence of $T\{\hat{a}_n(T)\}$, and we write this: $\mathcal{H}(T) = \{f(n)\}$.

Now, among transformations preserving finite measures normalised asymptotic type boils down to Kolmogorov–Sinai entropy, since

$$\mu(X) < \infty \Rightarrow \mathcal{H}(T) = \{h(T)n\}. \tag{4.15}$$

(To see this, note that $a_n(X, T)/\mu(X)^2$ is a return sequence, and hence any $\hat{a}_n(T) \sim \underline{h}\,(T) \cdot a_n(X, T)/\mu(X)^2 = \mu(X)h(T) \cdot \mu(X)n/\mu(X)^2 \doteq h(T)n$ as $n \to \infty$.)

We now show that the normalised asymptotic types of weakly isomorphic r.e.m.p.t.s in $\mathscr{C}_+$ coincide.

THEOREM 4.8. *Let T and T' be r.e.m.p.t.s s.t. $T, T' \in \mathscr{C}_+$ then $T \approx T'$ (i.e. T is weakly isomorphic with T') $\Rightarrow \mathcal{H}(T') = \mathcal{H}(T)$.*

PROOF. From (2.5), we have that (since $T \approx T'$ are r.e.m.p.t.s) $\exists c \in (0, \infty)$ s.t. $T \xrightarrow{c} T'$ and $T' \xrightarrow{1/c} T$. Hence an application of (2.3) gives

$$\frac{a_n(T')}{a_n(T)} \to c \quad \text{as} \quad n \to \infty \tag{4.16}$$

and a double application of (4.6) shows that

$$\underline{h}\,(T) = c\,\underline{h}\,(T'). \tag{4.17}$$

Combining (4.16) and (4.17) it is evident that $\hat{a}_n(T')/\hat{a}_n(T) \to 1$ as $n \to \infty$, i.e. $\mathcal{H}(T) = \mathcal{H}(T')$. Q.E.D.

We note that it follows from the above proof that

$$\begin{aligned} &T \approx T' (T, T' \textit{ r.e.m.p.t.s in } \mathscr{C}_+) \Rightarrow T \xrightarrow{\underline{h}(T)/\underline{h}(T')} T' \\ &\textit{and } T' \xrightarrow{\underline{h}(T')/\underline{h}(T)} T. \end{aligned} \tag{4.18}$$

Now let T_P be an ergodic Markov shift in $\mathscr{E}_+$ with transition matrix P, and let $s \in S$ be a state. Then, by (4.7) and (3.1),

$$\mathscr{H}(T_P) = \left\{ h_s(P) \sum_{k=0}^{n} p_{s,s}^{(k)} \right\}. \tag{4.19}$$

In particular, if $\underline{u}$ is a recurrent renewal sequence s.t. $T_{\underline{u}} \in \mathscr{E}_+$, then, by (4.8),

$$\mathscr{H}(T_{\underline{u}}) = \left\{ H(f(\underline{u})) \sum_{k=0}^{n} u_k \right\}. \tag{4.20}$$

We now construct uncountably many Markov shifts $\{T_\alpha\}_{0<\alpha<\frac{1}{2}}$ with the asymptotic type but distinct normalised asymptotic types.

Let $u_n = 1/\sqrt{n+1}$, then $\underline{u} = \{u_n\}_{n=0}^{\infty}$ is a recurrent renewal sequence, and (by Proposition 4.2) $T_{\underline{u}} \in \mathscr{E}_+$.

For $\alpha \in (0, \frac{1}{2})$, let $f_1(\alpha) = \alpha$ and $f_n(\alpha) = (1-\alpha) f_{n-1}(\underline{u})$ $(n \geqq 2)$. Denote by $\underline{u}(\alpha) = \{u_n(\alpha)\}_{n=0}^{\infty}$ the renewal sequence associated with $\{f_n(\alpha)\}$ and let

$$u_\alpha(\lambda) = \sum_{n=0}^{\infty} u_n(\alpha) \lambda^n \qquad 0 < \lambda < 1, \quad \text{and } T_\alpha = T_{\underline{u}(\alpha)}.$$

Now

$$H(f(\alpha)) = \eta(\alpha) + (1-\alpha) H(f) \tag{4.21}$$

where $\eta(\alpha) = \alpha \log(1/\alpha) + (1-\alpha)\log(1/(1-\alpha))$. Also

$$\frac{u(\lambda)}{u_\alpha(\lambda)} = (1-\lambda) u(\lambda) + \lambda(1-\alpha) \tag{4.22}$$

$\rightarrow (1-\alpha)$ as $\lambda \uparrow 1^-$. Hence (using Karamata's theorem [15], [6])

$$\frac{\sum_{k=1}^{n} u_k(\alpha)}{\sum_{k=1}^{n} u_k} \rightarrow \frac{1}{1-\alpha} \qquad \text{as} \quad n \rightarrow \infty. \tag{4.23}$$

Substituting (4.21) and (4.23) into (4.20)

$$\mathscr{H}(T_\alpha) = \left\{ \left(\left(\frac{\eta(\alpha)}{1-\alpha} \right) + H(f(\underline{u})) \right) \sum_{k=0}^{n} u_k \right\} = \left\{ 2 \left(\frac{\eta(\alpha)}{(1-\alpha)} + H(f) \right) \sqrt{n} \right\} \tag{4.24}$$

and we see that

$$\mathscr{A}(T_\alpha) = \{\sqrt{n}\} \qquad \forall \alpha \in (0, \tfrac{1}{2}). \tag{4.25}$$

Thus, from (4.24), it follows that

(4.26) $$\mathcal{H}(T_\alpha) = \mathcal{H}(T_\beta) \Leftrightarrow \frac{\eta(\alpha)}{1-\alpha} = \frac{\eta(\beta)}{1-\beta} \Leftrightarrow \alpha = \beta$$

$\because$ $\eta(\alpha)/(1-\alpha)$ increases strictly on $(0,\frac{1}{2})$. So $\{T_\alpha\}$ all have the same asymptotic type, but distinct normalised asymptotic types, hence no two distinct T_α and T_β can be weakly isomorphic.

§5. Random walks on Z

Let

(5.1) $$f = \{f_n\}_{n=-\infty}^{\infty} \text{ be s.t. } f_n \geqq 0 \ \forall n \in \mathbf{Z} \text{ and } \sum_{n=-\infty}^{\infty} f_n = 1.$$

Define the stochastic matrix $p_{s,t}(f) = f_{t-s}$ $\forall s, t \in \mathbf{Z}$. Clearly $m_s = 1$ $\forall s \in \mathbf{Z}$ defines a stationary distribution for $P(f)$.

We denote by T_f the Markov shift of the matrix $P(f)$ w.r.t. this stationary distribution and call it *the random walk with jump distribution f.*

If f satisfies (5.1) and $\Sigma_{n=-\infty}^{\infty} |n| f_n < \infty$ then (by Theorem 3.1 and [24] p. 33)

(5.2) $$T_f \textit{ is ergodic iff } \sum_{n=-\infty}^{\infty} nf_n = 0 \textit{ and } \text{g.c.d}\{n: f_n > 0\} = 1.$$

Let T_f be an ergodic random walk. An inspection of (4.7) shows that $\underline{h}(T_f) = \infty$, and so §4 does not apply to random walks.

Let

$$\sigma(f) = \left(\sum_{n=-\alpha}^{\infty} n^2 f_n\right)^{1/2} \quad \text{and} \quad \phi(t) = \sum_{n=-\infty}^{\infty} f_n e^{int}.$$

We shall begin by considering ergodic random walks with finite jump variance (i.e. f satisfies (5.2) and $\sigma(f) < \infty$).

THEOREM 5.1. *Let f satisfy* $\sigma(f) < \infty$ *and* (5.2), *then*

(5.3) $$\sum_{k=0}^{n} p_{0,0}^{(k)}(f) \sim \sqrt{\frac{2n}{\pi}} \frac{1}{\sigma(f)} \quad \textit{as} \quad n \to \infty.$$

PROOF. If $P(f)$ is aperiodic, then, by Spitzer's theorem ([24], p. 75),

(5.4) $$p_{0,0}^{(k)}(f) \sim \frac{1}{\sqrt{2\pi n}\,\sigma(f)} \quad \text{as} \quad n \to \infty.$$

If $P(f)$ has period $d > 1$ then $p_{0,0}^{(k)}(f) = 0$ when $d \nmid k$ and it can be shown from (5.4) that

(5.5)
$$p_{0,0}^{(kd)}(f) \sim \sqrt{\frac{d}{2\pi k}}\frac{1}{\sigma(f)} \quad \text{as} \quad k \to \infty.$$

From (5.5) follows (5.3). Q.E.D.

Now assume that f and f' satisfy (5.2) and that $\sigma(f), \sigma(f') < \infty$. If $T_f \xrightarrow{c} T_{f'}$ then

$$c = \lim_{n\to\infty}\frac{a_n(T_{f'})}{a_n(T_f)} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n} p_{0,0}^{(k)}(f')}{\sum_{k=0}^{n} p_{0,0}^{(k)}(f)} = \frac{\sigma(f)}{\sigma(f')} \quad \text{by} \quad (5.3)$$

i.e.

(5.6)
$$T_f \to T_{f'} \Rightarrow T_f \xrightarrow{\sigma(f)/\sigma(f')} T_{f'}$$
$$\textit{and} \quad T_f \not\xrightarrow{c} T_{f'} \quad \textit{for} \quad c \neq \frac{\sigma(f)}{\sigma(f')}.$$

We also see that

(5.7)
$$\sigma(f) < \infty \Rightarrow \mathcal{A}(T_f) = \{\sqrt{n}\}.$$

We now prove the converse to (5.7).

The following two lemmas slightly sharpen §6 on p. 72 of [24] (when $d = 1$).

LEMMA 5.2. *Let* $f_n \geqq 0$, $\Sigma_{n=-\infty}^{\infty} f_n = 1$ *and* $f_n = f_{-n}\ \forall n \in \mathbf{Z}$,

$$\text{g.c.d.}\{n : f_n > 0\} = 1,$$

$$\sigma(f) = \infty \Rightarrow \sqrt{n}\,p_{0,0}^{(n)}(f) \to 0 \quad \textit{as} \quad n \to \infty.$$

PROOF. Let $a(t) = (1 - \phi(t))/t^2$ (a real valued function). The following (easy) facts are proved in Spitzer ([24]):

(i) $p_{0,0}^{(n)}(f) = \int_0^{\pi} \phi(t)^n \frac{dt}{\pi}$,

(ii) $\inf_{t\in[-\pi,\pi]} a(t) = a_0 > 0$.

We note that also:

$$\text{(iii)} \quad \lim_{t\to 0} a(t) = \underline{\lim}_{t\to 0} \sum_{n=-\infty}^{\infty} f_n \frac{1-\cos nt}{t^2} \qquad \because f_n = f_{-n}$$
$$\geqq \sum_{n=-\infty}^{\infty} f_n \underline{\lim}_{t\to 0} \frac{1-\cos nt}{t^2} \quad \text{by Fatou's lemma}$$
$$= \tfrac{1}{2}\sigma(f)^2 = \infty.$$

Now, by (i)

$$\pi\sqrt{n}p_{0,0}^{(n)}(f)=\sqrt{n}\int_0^{\pi}(1-t^2a(t))^n dt$$

$$=\int_0^{\infty}\chi_{(0,\pi\sqrt{n})}(t)\left(1-\frac{t^2}{n}a\left(\frac{t}{\sqrt{n}}\right)\right)^n dt \quad \text{(changing variables)}$$

$$\leqq\int_0^{\infty}\chi_{(0,\pi\sqrt{n})}(t)e^{-a(t/\sqrt{n})t^2}dt\xrightarrow[n\to\infty]{}0$$

by dominated convergence since

$$\chi_{(0,\pi\sqrt{n})}(t)e^{-a(t/\sqrt{n})t^2}\leqq e^{-a_0t^2}\in L^1(\mathbf{R}_+)$$

by (ii) and

$$\chi_{(0,\pi\sqrt{n})}(t)e^{-a(t/\sqrt{n})t^2}\xrightarrow[n\to\infty]{}0 \qquad \forall t\in\mathbf{R}_+. \qquad \text{Q.E.D.}$$

LEMMA 5.3. *Let* $f_n\geqq 0$, $\Sigma_{n=-\infty}^{\infty}f_n=1$, g.c.d. $\{n: f_n>0\}=1$, *then*

$$\sigma(f)=\infty\Rightarrow\sqrt{n}p_{0,0}^{(n)}(f)\to 0 \quad as \quad n\to\infty.$$

PROOF. No generality is lost in assuming $P(f)$ to be aperiodic.

Let $g_n=\Sigma_{m=-\infty}^{\infty}f_mf_{m-n}$, then $g=\{g_n\}_{n=-\infty}^{\infty}$ satisfies the conditions of Lemma 5.2 and $\sigma(g)=\infty$. Thus

$$\sqrt{n}p_{0,0}^{(n)}(g)\to 0 \quad \text{as} \quad n\to\infty. \tag{5.8}$$

Also

$$\sum_{n=-\infty}^{\infty}g_ne^{int}=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}f_mf_{m-n}e^{int}=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}f_me^{imt}f_{m-n}e^{-i(m-n)t}$$
$$=|\phi(t)|^2. \tag{5.9}$$

So $\forall k\in\mathbf{Z}$, $n\geqq 1$

$$p_{0,k}^{(2n)}(f)=\int_{-\pi}^{\pi}\phi(t)^{2n}e^{-ikt}\frac{dt}{2\pi}\leqq\int_{-\pi}^{\pi}|\phi(t)|^{2n}\frac{dt}{2\pi}=p_{0,0}^{(n)}(g). \tag{5.10}$$

Thus by (5.10) and (5.8)

$$\sqrt{n}p_{0,k}^{(2n)}(f)\to 0 \quad \text{uniformly in } k\in\mathbf{Z} \quad \text{as} \quad n\to\infty. \tag{5.11}$$

Now let $\varepsilon>0$ and N s.t. $\forall n\geqq N$

$$\sqrt{n}p_{0,k}^{(2n)}(f)<\varepsilon \qquad \forall k\in\mathbf{Z}$$

then, if $n \geqq N$,

$$\sqrt{n}p_{0,0}^{(2n+1)}(f) = \sqrt{n} \sum_{k \in \mathbf{Z}} p_{0,k}^{(2n)}(f) f_{-k} \leqq \varepsilon.$$

Hence $\sqrt{n}p_{0,0}^{(n)}(f) \to 0$ as $n \to \infty$. Q.E.D.

THEOREM 5.4. *Let T_f and $T_{f'}$ be ergodic random walks with jump distributions f and f' respectively.*

If $\mathcal{A}(T_f) = \mathcal{A}(T_{f'})$ then $\sigma(f)$ and $\sigma(f')$ are finite or infinite simultaneously.

PROOF. By (5.7), it is sufficient to show that if $\sigma(f) = \infty$ then $\sqrt{n}p_{0,0}^{(n)}(f) \to 0$ as $n \to \infty$. But this is exactly Lemma 5.3. Q.E.D.

The above theorem is the converse to (5.7) (i.e. $\mathcal{A}(T_f) = \{\sqrt{n}\} \Rightarrow \sigma(f) < \infty$) and means that the only ergodic random walks similar to an ergodic random walk with jump distribution of finite variance can be other random walks with jump distribution of finite variance i.e. $\sigma(f) < \infty$, $T_f \sim T_{f'} \Rightarrow \sigma(f') < \infty$.

There is no analogue to (5.7) when $\sigma(f) = \infty$. To illustrate this, we construct an uncountable collection of ergodic random walks, with distinct asymptotic types, and hence pairwise dissimilar.

For $\alpha \in (1,2)$ let

$$f_n(\alpha) = \begin{cases} \dfrac{A(\alpha)}{|n|^{1+\alpha}} & |n| > 1 \\ 0 & n = 0. \end{cases} \qquad \text{where} \quad A(\alpha) = \frac{1}{2\zeta(1+\alpha)}$$

Then $\forall \alpha$: $\{f_n(\alpha)\}_{n \in \mathbf{Z}}$ satisfies (5.2).

Let T_α denote the (ergodic) random walk with jump distribution $\{f_n(\alpha)\}_n$. We calculate $\mathcal{A}(T_\alpha)$. Let

$$\phi_\alpha(t) = \sum_{n=-\infty}^{\infty} f_n(\alpha) e^{int} = \frac{1}{\zeta(1+\alpha)} \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \cos nt.$$

We begin by investigating the behaviour of $\phi_\alpha(t)$ near 0.

LEMMA 5.5 ([2], p. 141 ff.) $\forall \alpha \in (1,2)$ $\exists K_\alpha \in (0,\infty)$ *s.t.*

$$1 - \phi_\alpha(t) \sim K_\alpha |t|^\alpha \quad \textit{as} \quad t \downarrow 0. \tag{5.12}$$

PROOF. Recall that

$$\frac{1}{n^{1+\alpha}} = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty y^\alpha e^{-ny} dy \qquad \forall n \geqq 1.$$

Now, since

$$1-\phi_\alpha(t)=\frac{1}{\zeta(1+\alpha)}\sum_{n=1}^{\infty}\frac{1-\cos nt}{n^{1+\alpha}},$$

it is sufficient to show that

$$\psi_\alpha(t)=\sum_{n=0}^{\infty}(1-\cos nt)\int_0^\infty y^\alpha e^{-ny}dy\sim K_\alpha|t|^\alpha \quad \text{as} \quad t\downarrow 0. \tag{5.13}$$

Exchanging integral and summation signs and summing we see that

$$\psi_\alpha(t)=d(t)\int_0^\infty H_t(y)\,dy \tag{5.14}$$

where $H_t(y)=y^\alpha e^{-y}(1+e^{-y})/(1-e^{-y})((1-e^{-y})^2+2e^{-y}d(t))$ and $d(t)=1-\cos t$.

We note that $H_t(y)\uparrow H_0(t)$ as $t\to 0$ and that

$$\forall\varepsilon>0\qquad \int_0^\varepsilon H_0(y)\,dy=\infty,\qquad \int_\varepsilon^\infty H_0(y)\,dy<\infty. \tag{5.15}$$

From (5.15) it follows that if $G_t(y)\sim H_t(y)$ uniformly in $t>0$ as $y\to 0$ then

$$\int_0^\infty G_t(y)\,dy\sim\int_0^\infty H_t(y)\,dy \qquad \text{as} \quad t\to 0. \tag{5.16}$$

Now $H_t(x)\sim y^{\alpha-1}/(y^2+2d(t))$ uniformly in t as $y\to 0$ and so, by (5.14) and (5.16),

$$\begin{aligned}\psi_\alpha(t)&\sim d(t)\int_0^\infty\frac{y^{\alpha-1}dy}{y^2+2d(t)} && \text{as} \quad t\to 0\\ &=(d(t))^{\alpha/2}\int_0^\infty\frac{x^{\alpha-1}dx}{x^2+2} && \text{changing variables}\\ &\sim K_\alpha|t|^\alpha && \text{as} \quad t\to 0\end{aligned}$$

since $0<\int_0^\infty x^{\alpha-1}/(x^2+2)\,dx=2^{\alpha/2}K_\alpha<\infty$ and $d(t)\sim t^2/2$ as $t\to 0$. Q.E.D.

THEOREM 5.6. $\forall\alpha\in(1,2)$, $\mathscr{A}(T_\alpha)=\{n^{1-1/\alpha}\}$.

PROOF. We prove that $p_{0,0}^{(n)}(\alpha)\sim Kn^{-1/\alpha}$ as $n\to\infty$.

Let $\varepsilon>0$, by Lemma 5.5 $\exists\delta>0$ s.t.

$$1-(1+\varepsilon)K_\alpha|t|^\alpha\leqq\phi_\alpha(t)\leqq 1-(1-\varepsilon)K_\alpha|t|^\alpha \qquad \forall|t|\leqq\delta. \tag{5.17}$$

Now, arguing as in the proof of Spitzer's theorem ([24], p. 75)

$$\pi n^{1/\alpha} p_{0,0}^{(n)}(\alpha) = n^{1/\alpha} \int_0^\delta \phi_\alpha(t)^n dt + n^{1/\alpha} \int_\delta^\pi \phi_\alpha(t)^n dt. \tag{5.18}$$

Now

$$n^{1/\alpha} \int_\delta^\pi \phi_\alpha(t)^n dt \leqq (\pi - \delta) n^\alpha \Big(\sup_{|t| \geqq \delta} \phi_\alpha(t) \Big)^n \xrightarrow[n \to \infty]{} 0. \tag{5.19}$$

So, by (5.17),

$$\begin{aligned} \varliminf_{n\to\infty} n^{1/\alpha} \int_0^\delta (1-(1+\varepsilon)K_\alpha |t|^\alpha)^n dt &\leqq \varliminf_{n\to\infty} \pi n^{1/\alpha} p_{0,0}^{(n)}(\alpha) \leqq \varlimsup_{n\to\infty} \pi n^{1/\alpha} p_{0,0}^{(n)}(\alpha) \\ &\leqq \varlimsup_{n\to\infty} n^{1/\alpha} \int_0^\delta (1-(1-\varepsilon)K_\alpha |t|^\alpha)^n dt. \end{aligned} \tag{5.20}$$

Now, by a change of variables, it follows that $\forall c > 0$

$$n^{1/\alpha} \int_0^\delta (1 - c|t|^\alpha)^n dt = \int_0^{\delta n^{1/\alpha}} \left(1 - \frac{c|t|^\alpha}{n}\right)^n dt \xrightarrow[n\to\infty]{} \int_0^\infty e^{-c|t|^\alpha} dt = F(c)$$

by the dominated convergence theorem. $F(c)$ is a continuous function of c. Combining this fact with (5.20):

$$\pi n^{1/\alpha} p_{0,0}^{(n)}(\alpha) \to \int_0^\infty e^{-K_\alpha |t|^\alpha} dt \quad \text{as} \quad n \to \infty \qquad \text{Q.E.D.}$$

§6. Other ratio limit properties of e.m.p.t.s

In this section, the ratio limit properties of e.m.p.t.s are investigated further.

THEOREM 6.1. *Let $(X, \mathcal{B}, \mu, T)$ be an e.m.p.t. with $(X, \mathcal{B}, \mu)$ separable. Then $\forall A \in \mathcal{F}$ $\exists$ an algebra of subsets of A, a_A, μ-dense in $\mathcal{B} \cap A = \{B \in \mathcal{B}, B \subseteq A\}$ s.t.*

$$\frac{1}{a_n(A)} \sum_{k=0}^n \mu(B \cap T^{-k}C) \to \frac{\mu(B)\mu(C)}{\mu(A)^2} \qquad \forall B, C \in a_A.$$

PROOF. (Compare this proof to the proof of Theorem 1.4.)

Let $A \in \mathcal{F}$. In [11] and [19], it is proven that there is a measurable, compact, separable, completely disconnected topology on A, μ-dense in $\mathcal{B} \cap A$, such that T_A: $A \to A$ is a uniquely ergodic homeomorphism with unique invariant probability μ_A. Let a_A be the algebra of clopen subsets of A — it is clearly μ-dense in A. Let $\phi_n = (1/a_n(A))\sum_{k=0}^n \chi_A \circ T^k$ and $P_n(B) = \int_B \phi_n d\mu$: then $\{P_n\}_{n=1}^\infty \subseteq C(A)_+^*$ and $\|P_n\| = 1$ $\forall n$. Since A is compact and separable, every subsequence of $\{P_n\}_{n=1}^\infty$

has a weak $*$ convergent subsequence. By Lemma 1.3, all the weak $*$ limits of these subsequences are T_A-invariant, and hence, by unique ergodicity and the fact that $P_n(A) = 1\ \forall n$, equal to μ_A. In other words every subsequence of $\{P_n\}$ has a subsequence converging weak $*$ to μ_A —i.e. $P_n \xrightarrow[n\to\infty]{} \mu_A$ weak $*$. In particular

$$\frac{1}{a_n(n)}\sum_{k=1}^{n} \mu(B \cap T^{-k}A) \xrightarrow[n\to\infty]{} \frac{\mu(B)}{\mu(A)} \qquad \forall B \in a_A$$

and by symmetry (since $a_n(A, T) = a_n(A, T^{-1})$

$$\frac{1}{a_n(A)}\sum_{k=1}^{n} \mu(A \cap T^{-k}B) \xrightarrow[n\to\infty]{} \frac{\mu(B)}{\mu(A)} \qquad \forall B \in a_A.$$

Now choose $C \in a_A$ and let $\psi_n = (1/a_n(A))\sum_{k=1}^{n}\chi_C \circ T^k$ and $Q_n(B) = \int_B \psi_n d\mu$. Then $\{Q_n\} \subseteq C(A)_+^*$, $\|Q_n\| \leqq \|P_n\| \leqq 1\ \forall n$, and $Q_n(A) \to \mu(C)/\mu(A)$. An identical argument will show that $Q_n \xrightarrow[n\to\infty]{} \mu(C)/\mu(A)\, \mu_A$ weak $*$ Q.E.D.

In a similar manner to the proof of (i) $\Rightarrow$ (ii) of Proposition 1.1, a_A can be extended to a collection $\mathscr{C}_A$, μ-dense in $\mathscr{F}$, thus showing (1.4). Note that we have shown that (1.4) holds with a collection $\mathscr{C}_A$ which can be chosen to include any $A \in \mathscr{F}$. Theorem 6.2 shows that the collections $\{\mathscr{C}_A\}_{A\in\mathscr{F}}$ in fact form a very large non-homogeneous class when $\mu(X) = \infty$.

Theorem 6.2 also shows that in spite of Theorem 6.1, it is never true that $R(T) = \mathscr{F}$ when $\mu(X) = \infty$.

The existence of a similar result is mentioned in [7] (remark 4, p. 64). A stacking construction privately communicated by Krengel helped in the composition of Theorem 6.2.

THEOREM 6.2. *Let $(X, \mathscr{B}, \mu, T)$ be an e.m.p.t. with $\mu(X) = \infty$. Then*

$$\forall A \in \mathscr{F} \quad \exists B \in \mathscr{F} \quad \text{s.t.} \quad a_n(B)/a_n(A) \to \infty \quad \text{as} \quad n \to \infty.$$

To prove this theorem, a technical lemma is needed.

LEMMA 6.3. *Let $b_n, c_n > 0$, $\forall n$, be numbers s.t. $b_n \to \infty$, $c_n \geqq c_{n+1} \to 0$ and $\sum_{n=1}^{\infty} c_n = \infty$, then $\exists \{\varepsilon_n\}_{n=1}^{\infty}$ s.t.* (i) $\varepsilon_n \geqq \varepsilon_{n+1}$, (ii) $0 \leqq \varepsilon_n \leqq c_n$, (iii) $\sum_{n=1}^{\infty}\varepsilon_n < \infty$ *and* (iv) $b_n \sum_{k=n}^{\infty}\varepsilon_k \to \infty$ *as* $n \to \infty$.

PROOF. There is no loss of generality involved in assuming that $b_n \leqq b_{n+1}\ \forall n$ (for if not, work with $b_n^* = \inf_{k\geqq n}\{b_k\} \leqq b_{n+1}^* \leqq b_{n+1}\ \forall n$). Choose $\{n_k\}_k$ s.t. $\sum_k c_{n_k} < \infty$ and $\sum_k 1/\sqrt{b_{n_k}} < \infty$.

CLAIM. *$\exists \{m_k\}_{k=1}^{\infty}, \{\varepsilon_n\}_{n=1}^{\infty}$ such that $n_k \leqq m_k < m_{k+1}\ \forall k$, $0 \leqq \varepsilon_{n+1} \leqq \varepsilon_n \leqq c_n$, $\forall n$, and $b_{n_{k-1}}^{-1/2} \leqq \sum_{l=m_k+1}^{m_{k+1}} \varepsilon_l \leqq b_{n_{k-1}}^{-1/2} + c_{m_{k+1}}\ \forall k \geqq 2$.*

PROOF OF CLAIM. The kth inductive step is given.

Assume $m_1, \cdots m_k$ and $\varepsilon_1, \cdots \varepsilon_{m_k}$ have been chosen satisfying the conditions of the claim for $2 \leqq l \leqq k-1$. We show how to choose m_{k+1} and extend the sequence to $\varepsilon_1 \cdots, \varepsilon_{m_{k+1}}$.

Choose $M > m_k$, n_{k+1} s.t. $\delta = b_{n_{k-1}}^{-1/2}/(M - m_k) \leqq \varepsilon_{m_k}$ and for $n > m_k$ let

$$\delta_n = \begin{cases} \delta & \text{if} \quad \delta \leqq c_n \\ c_n & \text{else.} \end{cases}$$

Let $m_{k+1} = \min\{n > m_k : \sum_{j=m_k+1}^{n} \varepsilon_j \geqq b_{n_{k-1}}^{-1/2}\}$. It follows that $m_{k+1} \geqq M$ ($\because \delta_n \leqq \delta$ $\forall n$) and that $M < \infty$ ($\because \sum_n c_n = \infty$).

Let $\varepsilon_n = \delta_n$ for $m_k < n \leqq m_{k+1}$, then $0 \leqq \varepsilon_{n+1} \leqq \varepsilon_n \leqq c_n$ and

$$b_{n_{k-1}}^{-1/2} \leqq \sum_{j=m_k+1}^{m_{k+1}} \varepsilon_j \leqq b_{n_{k-1}}^{-1/2} + c_{m_{k+1}}.$$

Thus the claim is established by induction.

The sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ constructed has already been shown to satisfy (i) and (ii). It remains to prove (iii) and (iv).

(iii) $$\sum_{n=m_2+1}^{\infty} \varepsilon_n = \sum_{k=2}^{\infty} \sum_{j=m_k+1}^{m_{k+1}} \varepsilon_j \leqq \sum_{k=2}^{\infty} (b_{n_{k-1}}^{-1/2} + c_{m_{k+1}}) < \infty.$$

(iv) Let k_l be s.t. $n_{k_l-1} \leqq l < n_{k_l}$ then

$$\sum_{j=l}^{\infty} \varepsilon_j \geqq \sum_{j=m_k+1}^{m_{k+1}} \varepsilon_j \geqq b_{n_{k_l}-1}^{-1/2} \geqq b_l^{-1/2}.$$ Q.E.D.

PROOF OF THEOREM 6.2. Choose $A \in \mathcal{F}$, and for $n \geqq 1$ let

$$A_n = A \cap T^{-n}A - \bigcup_{k=1}^{n-1} T^{-k}A, \qquad B_n = \bigcup_{k=n+1}^{\infty} A_k = A - \bigcup_{k=1}^{n} T^{-k}A,$$

$$D_n = T^n B_n = T^n A - \bigcup_{k=0}^{n-1} T^k A, \qquad b_n = \frac{n}{a_n(A)} \quad \text{and} \quad c_n = \mu(B_n) = \mu(D_n).$$

It follows that $b_n \to \infty$, $c_n \downarrow 0$ and $\sum_{n=1}^{\infty} c_n = \infty$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of Lemma 6.3 appropriate to $\{b_n\}_n$ and $\{c_n\}_n$. Define $\{k_n\}_{n=1}^{\infty}$ by $c_{k_n} \leqq \varepsilon_n < c_{k_n-1}$, then $n \leqq k_n \leqq k_{n+1}$.

Using the non-atomicity of $(X, \mathcal{B}, \mu)$, one can find $F_n \subseteq A_{k_n}$ s.t. $\mu(F_n) = \varepsilon_n - c_{k_n}$. Let $E_n = \bigcup_{j=k_n+1}^{\infty} A_j \cup F_n$. Then $B_n \supseteq E_n \supseteq E_{n+1}$ and $\mu(E_n) = \varepsilon_n$ $\forall n$.

Let $B = \bigcup_{n=1}^{\infty} T^n E_n$. This is a disjoint union since $T^n E_n \subseteq D_n$ and $\{D_n\}_n$ are disjoint, so $\mu(B) = \sum_{n=1}^{\infty} \varepsilon_n < \infty$. Now

$$B \cap T^k B = B \cap \bigcup_{n=1}^{\infty} T^{(k+n)} E_n \supseteq \bigcup_{n=k+1}^{\infty} T^n E_n.$$

Hence

$$\frac{a_n(B)}{a_n(A)} \geqq \frac{1}{a_n(A)} \sum_{k=0}^{n} \sum_{j=k+1}^{\infty} \varepsilon_j \geqq \frac{n}{a_n(A)} \sum_{k=n}^{\infty} \varepsilon_k$$

$$= b_n \sum_{k=n}^{\infty} \varepsilon_k \to \infty \quad \text{as} \quad n \to \infty. \qquad \text{Q.E.D.}$$

References

1. L. M. Abramov, *On the entropy of a derived automorphism*, Dokl. Akad. Nauk SSSR **128** (1970), 647–650 (Russian); Amer. Math. Soc. Transl. (2) **49** (1965), 162–166 (English).

2. M. N. Barber and B. W. Ninham, *Random and Restricted Walks*, Gordon and Breach, New York, 1970.

3. P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, 1965.

4. K. L. Chung, *Markov Chains with Stationary Transition Probabilities*, Vol. 104, Springer, Heidelberg, 1960.

5. W. Feller, *Introduction to Probability Theory and its Applications*, Vol. 1, Wiley, New York, 1957.

6. W. Feller, *Introduction to Probability Theory and its Applications*, Vol. 2, Wiley, New York, 1966.

7. S. Foguel and M. Lin, *Some ratio limit theorems for Markov operators*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **23** (1972), 55–66.

8. A. B. Hajian and S. Kakutani, *Weakly wandering sets and invariant measures*, Trans. Amer. Math. Soc. **110** (1964), 136–151.

9. A. B. Hajian, Y. Ito and S. Kakutani, *Invariant measures and orbits of dissipative transformations*, Advances in Math. **9** (1972), 52–65.

10. G. Hansel and J. P. Raoult, *Ergodicity, uniformity and unique ergodicity*, Indiana Univ. Math. J. **23** (1973), 221–237.

11. T. E. Harris and H. Robbins, *Ergodic theory of Markov chains admitting an infinite invariant measure*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 860–864.

12. S. Kakutani, *Induced m.p.t.s*, Proc. Imp. Acad. Sci. Tokyo **19** (1943), 635–641.

13. S. Kakutani and W. Parry, *Infinite m.p.t.s with "mixing"*, Bull. Amer. Math. Soc. **69** (1963), 753–756.

14. T. Kaluza, *Über die Koeffizienten reziproker Potenzreihen*, Math. Z. **28** (1928), 161–170.

15. J. Karamata, *Sur un mode de croissance régulaire des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.

16. J. F. C. Kingman, *Regenerative Phenomena*, Wiley, New York, 1972.

17. E. M. Klimko, J. Yackel, *Entropy of first return partitions of a Markov chain*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **14** (1970), 251–253.

18. U. Krengel, *Entropy of conservative transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **7** (1967), 161–181.

19. W. Krieger, *On unique ergodicity*, Proc. Sixth Berkeley Symp. on Math. Stat. and Prob., 1970, pp 327–345.

20. W. Parry, *Ergodic and spectral analysis of certain m.p.t.s*, Bull. Amer. Math. Soc. **16** (1965), 960–966.

21. V. A. Rokhlin, *On the fundamental ideas of measure theory*, Mat. Sb. **25** (1949), 107–150 (Russian); Amer. Math. Soc. Transl. (1) **10** (1952), 1–54 (English).

22. V. A. Rokhlin, *On the decomposition of a dynamical system into transient components*, Mat. Sb. **25** (1949), 235–249 (Russian).

23. S. M. Rudolfer, *Some metric invariants for Markov shifts*. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **15** (1970), 202–207.

24. F. Spitzer, *Principles of Random Walk*, Van Nostrand, New York, 1964.

THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL